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ON CERTAIN PASSAGES IN JEVONS'S "THEORY OF POLITICAL ECONOMY."

THE diagrammatic method of studying economics may be regarded from three points of view.

(i.) Many teachers find in it a stimulating and helpful appeal to the eye, and use it as a short and telling way of making statements and registering results.

(ii.) A few students treat it as a potent instrument for giving precision to hypotheses in the first instance, and then for rigorously analyzing and investigating the results that flow from them.

(iii.) A very few investigators (among whom I think we must rank Jevons) have hoped ultimately to pass beyond the field of pure hypothesis and analysis, and to build up constructive results upon empirical curves of economic phenomena established by observation.

Precision and firmness in wielding the mathematical method as a hypothetical and analytical instrument are of the first importance; for, without them, all its other uses will turn out illusory. What may be called the "picturesque" use of diagrams, to illustrate theory, is fatally misleading unless an absolutely rigorous and precise interpretation is insisted on; and empirical or hypothetical data may be seriously misinterpreted, even by experts, for want of a sufficiently close preliminary analysis of the mathematical instrument of investigation.

I propose, then, to examine certain passages in Jevons's great work, in the hope of carrying his analysis a step further inward rather than of projecting his results further outward.

(i.) "Gregory King's" estimate of the variations in the price of wheat.

The celebrated estimate of the probable effect of serious deficiencies in the wheat harvest, usually attributed to Gregory King, but perhaps due to Davenant, is made the subject of an interesting investigation by Jevons.*

The estimate itself may be put into the following form: Taking 10 as the numerical value of the normal harvest and 1 as the normal price, it is estimated that we should have the price rising to 1.3 for a harvest of 9, to 1.8 for a harvest of 8, to 2.6 for a harvest of 7, to 3.8 for a harvest of 6, and to 5.5 for a harvest of 5.

Without inquiring into the grounds of this estimate, or making himself responsible for its correctness, Jevons tries to throw it into scientific form by deducing from it a law of price as a function of quantity. Taking the ordinates to register price, and the abscissas quantity, we shall have the following data: —

x . . . 10	9	8	7	6	5
y . . . 1	1.3	1.8	2.6	3.8	5.5

Of course there are an indefinite number of possible curves that pass through the six points thus indicated, and we shall be guided in our method of attack by any conceptions we may form on general grounds as to the probable form of the curve.

Jevons (p. 170) proceeds: "It is probable that the price of corn should never sink to zero, as, if abundant, it could be used for feeding horses, poultry, and cattle, or for other purposes for which it is too costly at present. It is said that in America corn, no doubt Indian corn, has been occasionally used as fuel. On the other hand, when the quantity is much diminished, the price should rise rapidly, and should become infinite before the quantity is zero, because famine would then be impending. The substitution of potatoes and other kinds of food renders the famine point very uncertain; but I think that a total deficiency of corn could not be made up by other food." These considerations lead Jevons to conjecture that the curve will be of the form $y = \frac{a}{(x-b)^n}$, and he fixes the constants so as to get a fair approximation to the values given in the estimate. He concludes, "Considering the close approximation in the above numbers, we may safely substitute the empirical formula for [Davenant's] numbers."

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* *Theory of Political Economy*, second edition, pp. 167–172. I take it that the estimate refers to wheat alone. See Davenant, *Essay upon the Probable Methods of making a people gainers in the Balance of Trade*, pp. 80, 81. But in one passage wheat, barley, and rye seem to be included.

Now I submit that, in the first place, the estimate, whether founded on observation or conjecture, obviously refers to wheat exclusively in its capacity as human food. Indeed, it is distinctly implied by Jevons that it is not actually used for any other purpose. If we are to consider its use as food for horses or (when burned) as manure, we shall have to take into account another curve, which will follow its own law, and will have to be added laterally to the curve we are now examining, as soon as the latter descends low enough to be affected by it.* But the fact that if wheat were cheaper people *would* buy it for horses, does not in any way, directly or indirectly, affect the price they really give, or the price they would give if the supply were *diminished*. Obviously, then, the law connecting the six points which constitute our data must be independent of such possible uses of wheat as are wholly inoperative throughout the region over which our observations (or conjectures) extend. In other words, our data belong to the curve that connects the price and the quantity of wheat *as human food*, and this curve will follow its own law independently of any other curves that may combine with it to form the total curve that gives the price of wheat as a function of its quantity. Now it is clear that a comparatively small increase of the supply of wheat would actually reduce its marginal degree of utility as human food to zero; that is to say, would give every potential purchaser as much as he wanted to eat. Our curve, then, must not, as Jevons thinks, be asymptotal to the axis of x, but must cut it for a comparatively low value of x.

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* This branch of the subject is well worked out by Walras in his "*Éléments d'Économie Politique Pure.* 11e Leçon, 30e Leçon *et passim*. Cf. my *Alphabet of Economic Science*, p. 60.

Again, impending famine will not make the price of wheat infinite. There is no such thing as an infinite price. Whether or not there can be an infinite *utility* is a question of some interest; and I am prepared to defend a negative answer even to that. But there can be no question at all as to the impossibility of an infinite *price*. It is a contradiction in terms. Again, a total failure of wheat, or even of grain in general, would no doubt produce famine, but not amongst the wealthy classes, and famine amongst poor people could not raise the price of corn to any very high figure: they can but offer all they have, and before the price of corn has risen many hundred per cent they will have no power to purchase it; their demand will cease to be "effective." Amongst the wealthy people and their retainers there will be no lack of meat and potatoes, vegetables, fruits, etc.; and wheat-bread, though commanding a high price, will not be purchased, in appreciable quantities, at what we are accustomed to think of as extreme famine prices, for there will be no famine amongst the purchasers, there will only be a lack of *bread* in the literal and narrow sense. I must therefore again join issue with Jevons in his second assumption; viz., that before we get back to the origin, our formula ought to give us an infinite value for y. Indeed, it is pitiable to think how slight the rise would probably have to be in order to induce incipient "famine," and how false the inference that if people are dying for want of a thing the price of that thing must be "infinite."

Divesting ourselves, then, of Jevons's preconceptions as to the general form of the curve, and reserving our own preconceptions (*viz.*, that the curve will cut both axes) to act as a check upon our results, let us look for the simplest law we can find which unites the six points. It will appear that they do not lie on a conic. The conic fixed by any five of them does not pass through the sixth. We next try a curve of the third degree. If we assume the simplest form, *viz.*:

$$y = ax^3 + bx^2 + cx + d,$$

we shall find that the curve determined by any four of the points passes through the other two.* Its formula will be

$$60y = 1500 - 374x + 33x^2 - x^3.$$

This curve cuts the axis of x between 13 and 14, and that of y at 25. These results have a *vraisemblance* which is truly remarkable when we consider how little right we have to expect such a curve as this to yield reasonable results when carried far beyond the limits of the data.

Such an outcome of our investigations can hardly fail to stimulate curiosity as to the origin of this most interesting estimate, and the grounds on which it was formed.

(ii.) Dimensions of economic quantities.

There are no portions of Professor Jevons's great work that are more difficult or (as I think) less satisfactory than the sections on the dimensions of economic quantities.

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* It may be conveniently found by the method of differences. Take four points:—

10	. . . .	1			
			3		
9	. . . .	1.3		2	
			5		1
8	. . . .	1.8		3	
			8		
7	. . . .	2.6			

It will be found that the law here suggested gives the other two points with perfect accuracy. I am indebted to Mr. John Bridge, of Hampstead, for suggesting the application of this method.

The previously uninitiated (of whom I am one) will be able to gather from the works of Professor Jevons himself that the theory of dimensions has been found a powerful instrument in the investigations of natural science, and will welcome his attempt to introduce the same method into economic studies. It is of vital consequence that we should have a precise conception of our several units and their relations to each other, if the mathematical method of economic study is to make any real progress; and the careful student will very rapidly learn to recognize in the theory of dimensions a valuable means of elucidating and checking his processes and results.

But the method, as applied by Jevons, appears to fit his diagrams singularly ill; and if it is to find any harmonious development in connection with them, some better principle of co-ordination must be sought.

Perhaps I shall be excused if I introduce the subject by a simple and elementary illustration of the theory itself, derived from the field of dynamics.

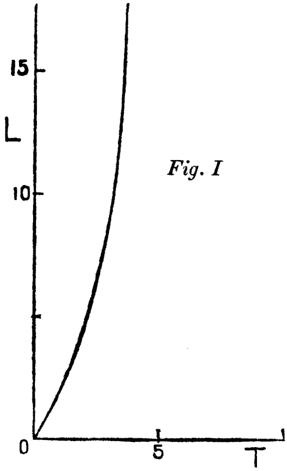


Fig. I

If we represent graphically the space which a body, falling from rest, traverses in any given time, under the action of gravitation, we shall have a curve roughly of the form of Fig. I., in which the ordinates represent length-space (10 feet to the unit), and the abscissas time (1 second to the unit). Here the dimension of the ordinates is *L*, or length, and that of the abscissas is *T*, or time. The *number* of the units contained in any ordinate is connected by a definite law with the *number* of the units contained in the corresponding abscissa ($s = 16\,t^2$, in feet and seconds), but the *nature* of the units in either case is entirely distinct. Thus the interpretation of an ordinate of a given length (when once obtained) is independent of the unit of time, because T does not enter as a dimension into the ordinates; but if I call a certain ordinate 10 when the unit of length is a foot, I must call it 120 if I change the unit to an inch. Again, if I call a certain abscissa 10 when the unit of time is a second, I must call it $\frac{10}{60}=\frac{1}{6}$ when the unit of time is a minute. Thus the numerical expression for any quantity of one positive dimension must be increased when the unit of its dimensions is decreased, and decreased when it is increased.

Let us now, treating the number of seconds in the formula $s=16t^2$ as the variable, and the feet traversed as the function, differentiate the latter to the former. That is to say, let us find the *rate* at which increments of time are increasing the space traversed, at any point in the course of the body; or, in other words, let us find the formula, and the curve, which will give us the *rate* at which the body is falling, as a function of the time it has been in motion.

The formula, of course, will be $v=32t$, and the curve is given roughly in Fig. II. Here the unit by which the abscissas are measured is the same as before. Their dimension is T. But the unit of the ordinates is no longer a unit of length. It is a unit of *rate*. An ordinate does not now represent feet, but feet-per-second. The unit of the new ordinates, then, is a unit of ratio between length and time, each measured in its own appropriate unit. Both L and T must therefore enter into the new ordinates as dimensions; but they do not enter upon the same footing.

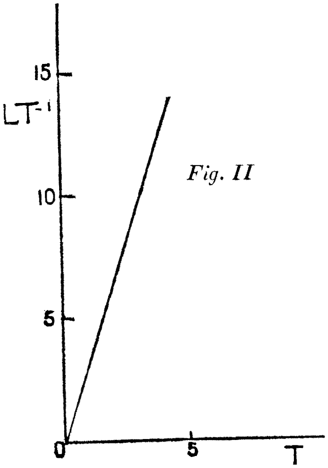
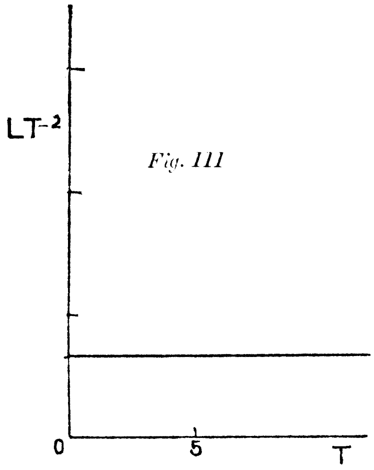


Fig. II

Sixteen feet a second is the same thing as 16 × 12 inches or 16 ÷ 3 yards per second; that is to say, L enters as a dimension into the new ordinates on the same footing as into the old ones. But 16 feet per second is 16 × 60 feet per minute; that is to say, T enters into the ordinates in the inverse relation of L. This is represented by saying that T enters negatively as a dimension. The dimensions of rate of movement will thus be LT^{-1}.

Since the process of differentiation always consists in establishing limiting *ratios* between increments of the independent variable and increments of the function, it will be clear at once from the above example that the dimensions of the variable must always enter negatively into the derived function, while the dimensions of the original function remain in the derived function positively.

Differentiating again, we shall obtain the rate at which the rate of motion is increasing, or what is usually called the acceleration of the falling body. It is given in Fig. III. The dimension of the abscissas is still T; but the variable of the function in Fig. II. having entered negatively once more into the ordinates, we shall now have $LT^{-1}T^{-1}$, or LT^{-2}, as the dimensions of the ordinates of Fig. III.



This indicates that if the unit of time be decreased, the numerical expression for acceleration must be decreased in the proportion of the *square* of the new unit to the *square* of the old unit. Thus an acceleration of 32 feet per second is an acceleration of 8 feet per half-second; *e.g.*, if gravitation is adding 32 feet per second to the velocity of a body, in each second, it is adding 16 feet per half-second, in each second, and 8 feet per half-second in each half-second. The new unit being half the old unit, the numerical expression for acceleration must be altered in the proportion of $(\frac{1}{2})^2 : 1^2$; *i.e.*, must be divided by 4.

Now note further that in these successive figures an *area* in one always represents the same kind of quantity, and has the same dimensions, as the *ordinate* of its predecessor.

Thus on Fig. III. if we take the area above the abscissa 2, or $\int_0^2 f''(x) . dx$, we shall, of course, have a quantity of the dimensions $LT^{-2}T$, or LT^{-1}; *i.e.*, a velocity. But the ordinates on Fig. II. are velocities. If, again, we take the area above the abscissa 2 in Fig. II., or $\int_0^2 f'(x) . dx$, we shall have a quantity of dimensions $LT^{-1}T$, or L; *i.e.*, a length. But the ordinates on Fig. I. represent lengths.

It follows that there is no natural or inherent propriety in representing each actual dimension of the quantity we may be dealing with, by a dimension of space in a diagram, for we have seen that length and velocity may either of them be represented with equal propriety by a line or an area. In the same way area or volume itself may often be suitably represented by a line in a diagram. Again, there is no impropriety or inconvenience in making diagrams in which the same dimension enters positively or negatively into two or more axes. Thus, in our Fig. II., T enters positively into the abscissas and negatively into the ordinates.

An apparent neglect of these considerations, which I am not able satisfactorily to explain, has, if I am not mistaken, introduced needless difficulty and obscurity into Jevons's investigations of the dimensions of economic quantities, and has robbed his results of lucidity, if it has not led him into positive error.

Instead of criticising in detail the passages in the *Theory of Political Economy*, in which this subject is treated, I will go over the ground which they cover, and ask the reader to compare my statements with those of Jevons.

We will begin with total utility. If we use capitals for dimensions and minuscules for the number of units (*e.g.*, T for the dimension time, and t for the number of seconds or other units of time), we may indicate the units of total utility resulting from any consumption of commodity by u, and the number of units in the corresponding amount of commodity by q. The fundamental quantitative fact with which Economics have to deal may then be expressed in the thesis that u is always a function of q.

Now Jevons shows that, q being the variable, the final degree of utility of a commodity is the differential coefficient to q of its total utility; whence it follows that, taking U as the dimension of total utility, and Q as the dimension of commodity, we shall have the dimensions of final degree of utility UQ^{-1}.

Jevons uses the symbol U to signify final degree of utility (*cf.* Jevons, p. 71), but I think this notation is calculated to mislead. I should suggest that when we wish to speak of final degree of utility without entering upon the analysis or history of the conception, we should indicate the number of units by v, and the dimension by V.

In comparing my formulæ with Jevons's, therefore, it must be borne in mind that his u corresponds to mine; his U as a dimension corresponds to my V or UQ^{-1}; his U as a quantity to my v, which will be the differential coefficient of u to the variable q.*

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* In substituting Q for M, I follow the indications of Jevons himself. Preface to 2d edition, p. xi. On page 71 Jevons appears to use contradictory and inconsistent language with regard to "intensity of feeling," which he identifies in one place with "degree of utility" and more correctly defines two lines above as total instantaneous utility. The former of these quantities has the dimensions UQ^{-1}, the latter $UQ^{-1}QT^{-1}$, or UT^{-1}. *Vide infra*, I am indebted to Mr. W. E. Johnson, of King's College, Cambridge, for the elucidation of this point.

Now final degree of utility determines exchange value, and we have: Exchange value determined by v (of dimensions UQ^{-1}); that is, by *rate at which increments of commodity are increasing total advantage derived from consumption.* In this sense the dimensions of "value in exchange" may be said to be UQ^{-1}.

Jevons prefers to regard total utility as a quantity of two dimensions, MU, corresponding to my QV, and final degree of utility as a quantity of one dimension, U, corresponding to my V. If we adopt this view, it would be proper to make final degree of utility our starting-point, and begin with v as a function of q. We should then *integrate* to obtain u, of dimensions QV. My objection to this is twofold; for total utility is susceptible of direct measurement by any standard of effort or endurance that may be selected (such as foot-tons of work done under assigned conditions), whereas final degree of utility* is essentially a (limiting) *ratio*, and is therefore appropriately represented (like all ratios) as having two dimensions (whether simple or complex, homogeneous or heterogeneous) which enter the one positively and the other negatively.

Thus, if we say with Jevons that total utility has two dimensions, MU (our QV), we must, I think, add that one of these dimensions, U (our V), is a ratio, and not properly a dimension at all. In our notation it is equivalent to UQ^{-1}, and the dimensions QUQ^{-1} reduce to U.

In my view, it does not at all follow from this that there is any impropriety in representing total utility diagrammatically by area.† We shall do so whenever we draw curves of quantity and final degree of utility. The dimensions of abscissas will be Q, of ordinates UQ^{-1}, and of areas QUQ^{-1}, or U.

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* It must be remembered that when we speak of the direct measurement of final degrees of utility or value it is not really these quantities we are measuring, but the product of final degree of utility into a small increment of commodity. It is not $\frac{du}{dq}$, but $\frac{du}{dq} \cdot dq$ or du; *i.e.*, a small increment, of dimension U, which we measure.

† Mr. Johnson informs me that writers on the Newtonian dynamics habitually represent linear space by area in their diagrams. This is obviously convenient.

But Jevons points out that as a matter of fact it is not supply but rate of supply per unit of time, not total enjoyment but rate of enjoyment, with which we are concerned. Whether this is universally true in any fruitful and manageable sense or not, it is certainly true of all such commodities as food, water, etc. We must therefore take up the question again from this point of view. Regarding rate of supply per unit of time (dimensions QT^{-1}) as the variable, and rate of enjoyment, relief, or advantage per unit of time (dimensions UT^{-1}) as the function, and then differentiating, we shall find that the dimensions T cancel each other, and we have $UT^{-1}Q^{-1}T$, or UQ^{-1} again, as the dimension of the *rate at which increase in rate of supply increases rate of enjoyment.* And it is, in truth, sufficiently plain that this *rate* is a direct relation between the quantity of the commodity and the enjoyment it causes, and is not affected in its numerical expression by any change in the unit of time.

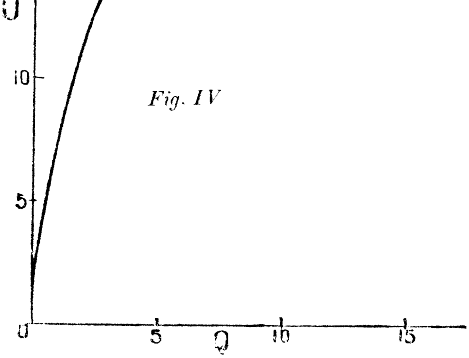

Fig. IV

These results are summarized on Figs. IV.–VII.: in Fig. IV. we have dimension of abscissa Q, and dimension of ordinate U; in Fig. V. of abscissa Q, of ordinate UQ^{-1}, of area U; in Fig. VI. of abscissa QT^{-1}, of ordinate UT^{-1}; in Fig. VII. of abscissa QT^{-1}, of ordinate UQ^{-1}, of area UT^{-1}; where the areas in Figs. V. and VII. have the same dimensions, respectively, as the ordinates in Figs. IV. and VI.

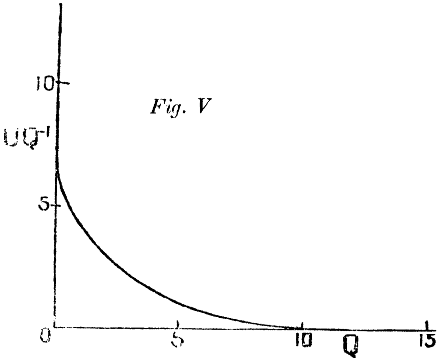
Fig. V

If we wished to represent, with the aid of Fig. VII., the total advantage derived from the consumption of a given quantity of commodity at the rate indicated, we should have to add a third axis perpendicular to the plane of the figure, on which to measure the time during which the rate of enjoyment represented by the area is maintained. Neither of Jevons's objections to this method are valid. There is no reason why an economic quantity of one or of two dimensions should not be represented by a figure of three dimensions; and there is no objection to introducing time positively on one axis and negatively on another.

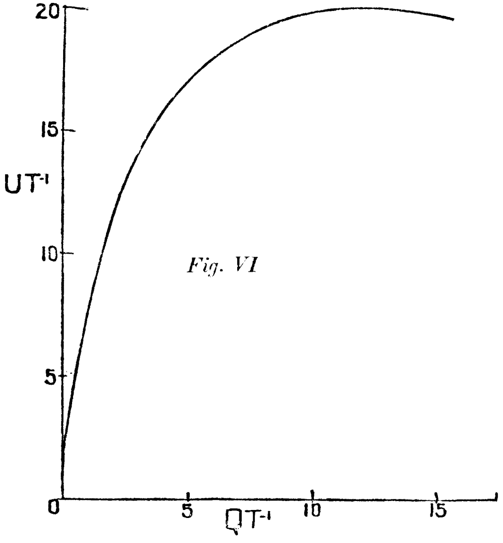
Fig. VI

It should be observed that this method renders a perfect account of the fact that (under ordinary circumstances, and with due limitations) we must hold that the same amount of commodity yields a larger sum of satisfaction when consumed slowly than when consumed fast. The result of slackening the rate of supply would be to shorten the abscissas in Figs. VI. and VII., and proportionately to lengthen the perpendicular time-axis in the solid figure built on Fig. VII. This would obviously increase the volume of the solid that represents the total utility.

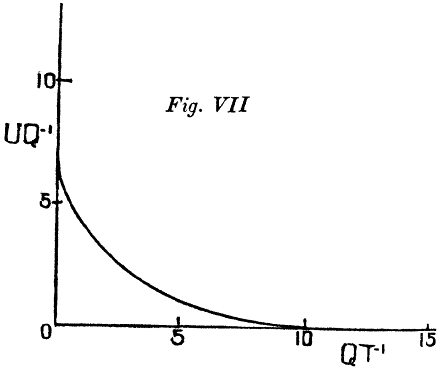

Fig. VII

Such a figure would represent all the quantities with which we have to deal. Rate of supply on the axis of X, dimensions QT^{-1}; final degree of utility on axis of Y, dimensions UQ^{-1}; time on perpendicular axis of Z, dimension T; rate of enjoyment on area of plane figure, dimensions $QT^{-1}UQ^{-1}$, or UT^{-1}; total enjoyment on volume of solid figure, dimensions $QT^{-1}UQ^{-1}T$, or U; total supply on rectangle between axis of X and axis of Z, dimensions $QT^{-1}T$, or Q.

Of these quantities, the rate of supply and the final degree of utility are the most important, and these are the most easily read on the figure.

We have now considered the case of absolute quantity of commodity yielding absolute quantity of enjoyment, and also the case of rate of supply of commodity yielding rate of enjoyment; but there is a third and equally important case, in which absolute quantity of commodity yields rate of enjoyment. Thus we are accustomed to think of furnished apartments as yielding so much advantage per week, month, or year, not as yielding a certain total advantage. The correctness, or at any rate the completeness, of this view may well be questioned, but in the case of imperishable articles, such as diamonds, it is difficult to regard the variable and function in any other light than that of absolute quantity and rate of advantage.

In the first place, then, we shall measure quantity of commodity, as the variable, along the axis of X, with dimension Q, and rate of enjoyment, as the function along the axis of Y, with dimensions UT^{-1}. Differentiating, we shall get the rate at which increments of commodity are increasing the rate of enjoyment, as a function of q with dimensions $UT^{-1}Q^{-1}$. This is not final degree of utility (dimensions UQ^{-1}), but a ratio between this quantity and time; and it is the measure, not of *value* and thence of *price*, but of *value-per-unit-of-time* and thence of *hire*.*

Obviously the problem of interest, or hire of capital, must fall under this general case. Capital is a commodity and is measured in absolute units, whereas the advantage of capital is a periodic yield and is measured by a ratio between time and commodity. The peculiarity of the case is that here the advantage itself consists in the obtaining of commodity, so that the dimension U will itself be Q. Thus in the case of capital the dimensions of hire $UT^{-1}Q^{-1}$ become $QT^{-1}Q^{-1}$, or T^{-1}. This must be the dimension of hire of capital (that is to say, of interest) considered as a rate; and we shall see presently that an independent investigation of the phenomena of interest leads to the same conclusion.

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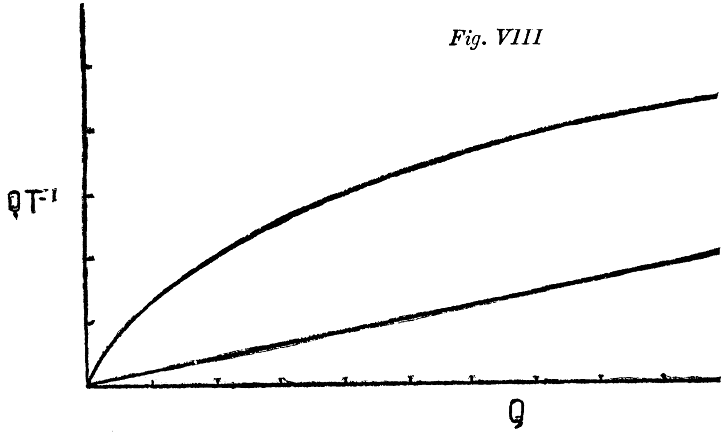
*Both price and hire to be understood as *per unit of commodity*. To establish a relation between hire and price, we must suppose the purchaser's estimate of distant enjoyment to be affected by uncertainty, or some other quality inherent in remoteness, in such a way as to make the successive anticipated yields of successive increments of time a convergent series. Price will then be the integral of *dt*. (hire), and will have the dimensions of hire and time; viz., $UT^{-1}Q^{-1}T$, or UQ^{-1}, as before.

Jevons's objection to representing identical or similar quantities now by one and now by another kind of geometric quantity, and to introducing the same economic dimension upon two axes, leads him to criticise with quite needless severity, as I take it, Peacock's observations on the subject of interest (*Theory*, pp. 271 *sq.*), and further to undervalue his own diagrammatic representation of the phenomena in question, and to obscure his own results.

The problem may be attacked thus: Suppose an industrial concern *in which a fixed amount of labor is employed* to command c units of capital, c being variable. Obviously we may treat the capital as commodity, with the single dimension Q. Now consider the rate per unit of time at which that capital will wear out and disappear. We must suppose the stock always to be replaced as fast as it disappears, and may take τ units of time (say years) as the period during which the whole will have been renewed. Measuring the annual wear not in percentage, but in absolute units of capital, we shall then have the annual wear equal to $\frac{c}{\tau}$, and its dimensions will be QT^{-1}. This quantity is a function of c. Probably τ itself will vary according to the amount of c; that is to say, the number of units in τ will be a function of the number of units in c; for we may suppose that for early increments of capital the annual wear will increase less than proportionately to the increase of the capital, but when the amount of capital becomes very large it will be difficult, with the fixed amount of labor at command, to look after it properly, and it will wear more rapidly. We may, however, neglect this consideration and assume that τ will be a constant, and $y = \frac{c}{\tau}$, the equation of a straight line. In Fig. VIII., then, the abscissas of points on the line OW might represent the quantity of capital, and the ordinates the amount of annual or other periodic wear, as a function of c. We have next to examine the productiveness of the capital; *i.e.*, the number of units of commodity, per annum or other unit



of time, which the use of the capital enables the fixed application of labor to produce. In estimating this we must subtract all the commodity which the capital actually consumes, such as coal, oil, etc. (supposing the capital to be in the shape of machinery).* We may take p as the amount of commodity which c enables the fixed application of labor to produce (over and above what c itself consumes) every year, or other period of time. Its dimensions will be QT^{-1}. Then $p\tau$ will be total amount of commodity produced in the time τ in virtue of the use of c. And if we put q for $p\tau$ then the *annual* product, or p, may be written $\frac{q}{\tau}$. Its dimensions will be the same as those of $\frac{c}{\tau}$; viz., QT^{-1}.

As the capital increases in amount, its annual yield, $\frac{q}{\tau}$, will at first increase rapidly, but after a time (the application of labor being fixed) increase in the amount of capital will but slightly increase the annual yield (since the labor is already supplied with its most urgently required aids), and at last, when the point has been reached at which the labor has all possible aids and is applied at the greatest possible advantage, further increments of capital will not increase the annual product at all. This quantity, $\frac{q}{T}$, having the same dimensions as $\frac{c}{T}$, may be shown on the ordinates of the same figure. It is given, hypothetically, on the upper curve of Fig. VIII., as a function of c. This *gross* productiveness of c, as we have seen, reaches a maximum, or at any rate has a limit; but long before it reaches it, the *net* productiveness of c will have passed its maximum and will be tending to zero; for we must remember that as c increases, the annual waste of c also increases without limit, and since this annual waste must be made good, the *net* productiveness of c is represented by $\frac{q}{T}-\frac{c}{T}$ (*i.e.*, annual yield of capital minus annual waste of capital) in the figure the length of the intercept between the two curves. Now this quantity, $\frac{q-c}{T}$, is itself a function of c, and has the dimensions QT^{-1}. It is represented in Fig. IX.

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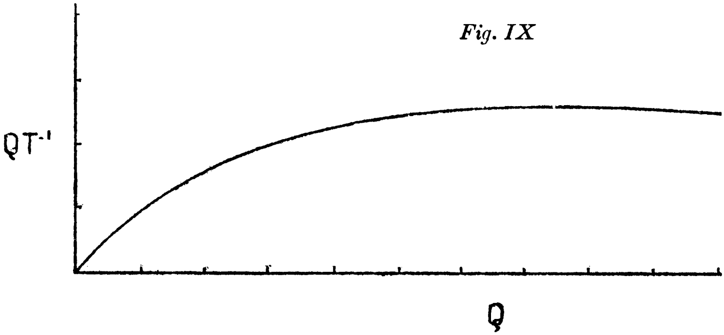
* I assume the existence of a common measure of all "commodities."



If we now proceed to differentiate, to the variable c, we are in danger of having to deal with ordinates so small as to defeat the object of diagrammatic illustration; but this may be averted by the familiar artifice of lengthening the scale on the axis of Y. We will therefore represent the unit of Q by the same length as before on the axis of X, and by a length ten times as great on the axis of Y. We shall then have a curve such as that of Fig. X., which will show us the *rate at which increments of capital are increasing the annual return made by the capital.*

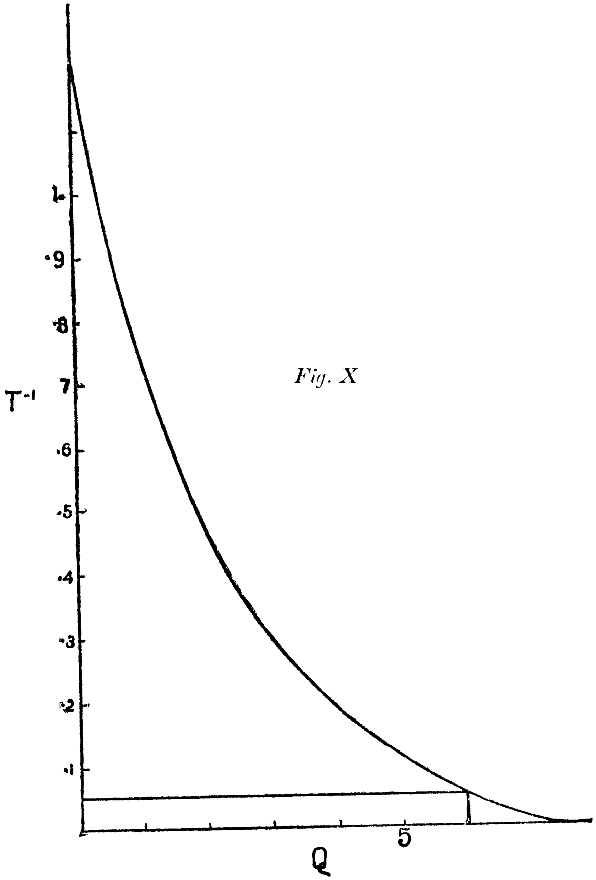


Fig. X

Now we have seen that the rate of hire of anything follows the ordinary laws of final degree of utility, and is determined by the rate of productiveness (in satisfaction or commodity) of the *last increment* of the thing hired. That is to say, if c is the quantity of capital and $f(c)$, in Fig. IX., the net periodical productiveness of c, then $f'(c)$, in Fig. X., will be the *rate of hire* of capital; *i.e.*, the *rate of interest.*

This quantity is a ratio between rate-of-supply-of-commodity (yield) and commodity (capital), and its dimensions therefore are $QT^{-1}Q^{-1}$, or T^{-1}; and Jevons has shown, with great care and elaborateness, that T^{-1} is in truth the dimension of rate of interest (*Theory*, etc., pp. 268 *sq.*). In fact, the length of any ordinate in Fig. X. shows, in numerical units, without dimension, the *ratio* between the increase of the capital and the increase of the periodical yield or product. For $x = 5\frac{1}{3}$ it is one-tenth, or ten per cent; for $x = 6\frac{1}{4}$ it is one-twentieth, or five per cent. All that we need to know more is the *length of the period*, for which the periodic yield has been estimated. That is to say, the only dimension of rate of productiveness, or rate of interest, is T^{-1}. The numerical expression of a given rate of interest is only affected by a change in the unit of time, not by a change in the unit of commodity.

Proceeding, then, with the examination of Fig. X., we find that $f'(c) \,.\, c$ is the actual sum periodically paid as interest; $\int_0^c f'(c) \,.\, dc$, or the total curvilinear area over c, the total net periodical yield of the given application of labor, backed by the quantity c of capital, and

$$\int_0^c f'(c) \,.\, dc - f'(c) \,.\, c,$$

or the curvilinear area over the rectangle of interest, the periodical return to the application of labor over and above the sum paid in interest. All these quantities have the dimensions QT^{-1}, and are periodical. To get the absolute sum of any one of them during a defined period τ, we should have to multiply by τ and reduce the dimensions to $QT^{-1}T$, or Q. This would involve a third axis, registering positively the dimension T, which appears negatively on the axis of Y. What are the grounds of Jevons's objection to this I have not been able to discover, and I am wholly unable to defend his position (*cf. Theory*, pp. 72 *sq.*).

If the view now set forth is correct, no great importance can be attached to the paragraph on pp. 266, 267 of the *Theory*, in which Jevons seeks a "general expression for the rate of interest." His fundamental hypothesis that the produce for the same amount of labor may reasonably be regarded as a continuous function of the time elapsing between the expenditure of the labor and the enjoyment of the result is not based upon a typical case of the use of capital, and in the cases to which it does apply it deals with derivative, not with primary facts and phenomena. The typical case of the use of capital is that in which the result is yielded continuously. All the great staple industries need a continuous renewal and expansion of capital, which capital, as it is invested, forthwith begins to yield a continuous product. This I take to be the primary and norm-giving fact. If, by way of exception, an investment of capital is proposed which will, after an interval, yield not a revenue, but an absolute utility; or if, as is extremely common, a gradual investment of capital is proposed, with the expectation that when the investment is complete the whole invested capital (in the shape of a ship or a machine, for instance) will be purchased by some one who has performed the process of integration indicated in the note on p. 13; or, lastly, if an immediate investment of capital is proposed in order that after an interval a periodic yield may be enjoyed by the investor, — in all these cases the investor has to consider what quantity of commodity he would command at the expiration of the given time, had he invested at first in one of the staple industries, and then continuously reinvested his continuously accruing return in the same industry again. If the proposed investment does not promise equal advantages, he will not enter upon it. Thus the basis of the estimate in every case of *deferred result* must be sought in the rate of *immediate yield* (*cf. Theory*, pp. 66–74, 90–91, 266–280).

In these notes I have made no attempt to carry the theory of capital and interest beyond the point at which Jevons left it. Very much remains to be done in this field, but my present object is only to clear away certain difficulties and rearrange the results already obtained, in order that the ground from which we are to advance may be better and more firmly occupied.

As an exercise we might trace the effect of any process which would make capital more durable. This would increase τ, and so lower the curve of wear in Fig. VIII. But it probably would not lower the curve of productiveness, since it would increase the numerator as well as the denominator of the formula that gives the ordinate, and that, too, in something like the same proportion. Hence the ordinates of Fig. IX. will be lengthened, and so will those of Fig. X. The immediate effect, therefore, if we could imagine the phenomenon taking place suddenly and simultaneously everywhere, would be to raise the rate of interest. But the increased net production would tend to increase accumulation, and so c would increase, and $f(c)$ and $f'(c)$ would decline again.

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